HYBRID SYNCHRONIZATION OF LIU AND LÜ CHAOTIC SYSTEMS VIA ADAPTIVE CONTROL

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ABSTRACT

This paper derives new results for the hybrid synchronization of identical Liu systems, identical Lü systems, and non-identical Liu and Lü systems via adaptive control method. Liu system (Liu et al. 2004) and Lü system (Lü and Chen, 2002) are important models of three-dimensional chaotic systems. Hybrid synchronization of the three-dimensional chaotic systems addressed in this paper is achieved through the synchronization of the first and last pairs of states and anti-synchronization of the middle pairs of the two systems. Adaptive control method is deployed in this paper for the general case when the system parameters are unknown. Sufficient conditions for hybrid synchronization of identical Liu systems, identical Lü systems and non-identical Liu and Lü systems are derived via adaptive control theory and Lyapunov stability theory. Since the Lyapunov exponents are not needed for these calculations, the adaptive control method is very effective and convenient for the hybrid synchronization of the chaotic systems addressed in this paper. Numerical simulations are shown to illustrate the effectiveness of the proposed synchronization schemes.

KEYWORDS

Adaptive Control, Chaos, Hybrid Synchronization, Liu System, Lü System, Synchronization.

1. INTRODUCTION

A chaotic system is a very special nonlinear dynamical system, which has several properties such as the sensitivity to initial conditions as well as an irregular, unpredictable behaviour. This sensitivity to initial conditions of chaotic systems is popularly called as the *butterfly effect* [1].
Chaos is an interesting nonlinear phenomenon and has been extensively studied in the last two decades [1-40]. Chaos theory has been applied in many scientific disciplines such as Mathematics, Computer Science, Microbiology, Biology, Ecology, Economics, Population Dynamics and Robotics. Especially, chaos synchronization has found important applications in areas such as secure communications, data encryption, etc.

In 1990, Pecora and Carroll [2] deployed control techniques to synchronize two identical chaotic systems and showed that it was possible for some chaotic systems to be completely synchronized. From then on, chaos synchronization has been widely explored in a variety of fields including

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physical systems [3-4], chemical systems [5-6], ecological systems [7], secure communications [8-10], etc.

In most of the chaos synchronization approaches, the *master-slave* or *drive-response* formalism is used. If a particular chaotic system is called the *master* or *drive* system and another chaotic system is called the *slave* or *response* system, then the idea of the synchronization is to use the output of the master system to control the slave system so that the output of the slave system tracks the output of the master system asymptotically.

Since the seminal work by Pecora and Carroll [3], a variety of impressive approaches have been proposed for the synchronization of chaotic systems such as the OGY method [11], active control method [12-16], adaptive control method [17-22], sampled-data feedback synchronization method [23], time-delay feedback method [24], backstepping method [25-26], sliding mode control method [27-32], etc.

So far, many types of synchronization phenomenon have been presented such as complete synchronization [3], phase synchronization [33], generalized synchronization [34], anti-synchronization [35-37], projective synchronization [38], generalized projective synchronization [39-40], etc.

Complete synchronization (CS) is characterized by the equality of state variables evolving in time, while anti-synchronization (AS) is characterized by the disappearance of the sum of relevant variables evolving in time.

Projective synchronization (PS) is characterized by the fact that the master and slave systems could be synchronized up to a scaling factor, whereas in generalized projective synchronization (GPS), the responses of the synchronized dynamical states synchronize up to a constant scaling matrix α.

It is easy to see that the complete synchronization (CS) and anti-synchronization (AS) are special cases of the generalized projective synchronization (GPS) where the scaling matrix $\alpha = I$ and $\alpha = -I$, respectively.

In hybrid synchronization of chaotic systems [41-42], one part of the system is synchronized and the other part is anti-synchronized so that the complete synchronization (CS) and anti-synchronization (AS) coexist in the system. The coexistence of CS and AS is highly useful in secure communication and chaotic encryption schemes.

In this paper, we investigate the hybrid chaos synchronization of uncertain three-dimensional chaotic systems, viz. identical Liu systems ([43], 2004), identical Lü systems ([44], 2002) and non-identical Liu and Lü systems. We consider the general case when the parameters of the systems are unknown.

This paper is organized as follows. In Section 2, we provide a description of the chaotic systems addressed in this paper, *viz.* Liu system (2005) and Lü system (2002). In Section 3, we discuss the hybrid synchronization of identical Liu systems via adaptive control. In Section 4, we discuss the hybrid synchronization of identical Lü systems via adaptive control. In Section 5, we discuss the hybrid chaos synchronization of identical Lü systems via adaptive control. Section 6 summarizes the main results obtained in this paper.

2. SYSTEMS DESCRIPTION

The Liu system ([43], 2004) is described by the Liu dynamics

$$
\begin{aligned}
\dot{x}_1 &= a(x_2 - x_1) \\
\dot{x}_2 &= bx_1 - x_1 x_3 \\
\dot{x}_3 &= -cx_3 + dx_1^2
\end{aligned}
\tag{1}
$$

where x_1, x_2, x_3 are the state variables and a, b, c, d are positive, constant parameters of the system.

The Liu system (1) is chaotic when the parameter values are taken as

$$a = 10, \quad b = 40, \quad c = 2.5 \quad \text{and} \quad d = 4$$

The state orbits of the Liu chaotic system (1) are shown in Figure 1.

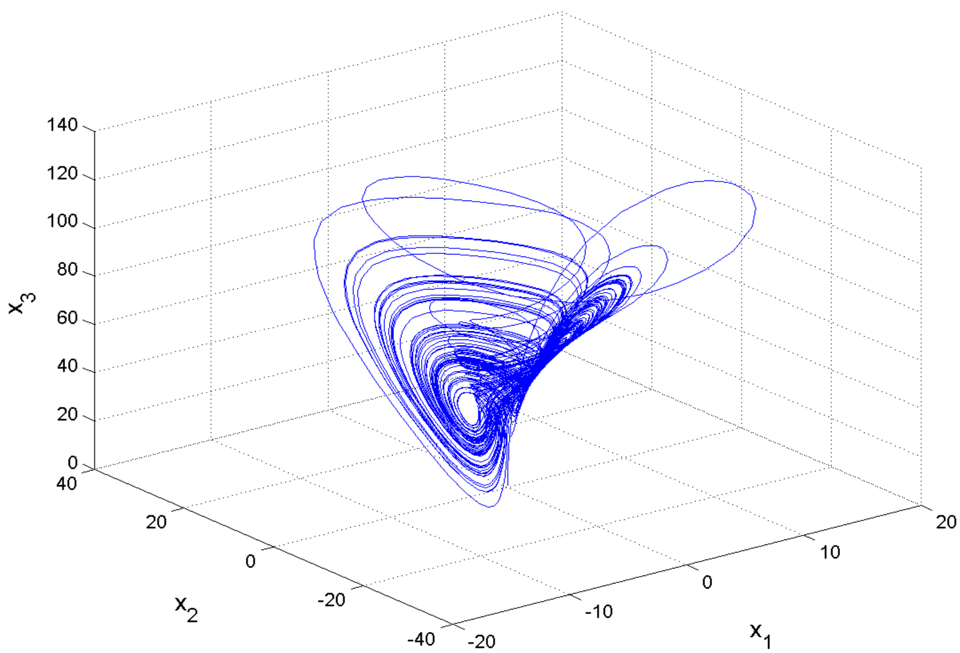


Figure 1. State Orbits of the Liu Chaotic System

The Lü system ([44], 2002) is described by

$$
\begin{aligned}
\dot{x}_1 &= \alpha(x_2 - x_1) \\
\dot{x}_2 &= \gamma x_2 - x_1 x_3 \\
\dot{x}_3 &= -\beta x_3 + x_1 x_2
\end{aligned}
\tag{2}
$$

where x_1, x_2, x_3 are the state variables and α, β, γ are positive constant parameters of the system.

The Lü system (2) is chaotic when the parameter values are taken as

$$\alpha = 36, \quad \beta = 3 \quad \text{and} \quad \gamma = 20$$

The state orbits of the Lü chaotic system (2) are shown in Figure 2.

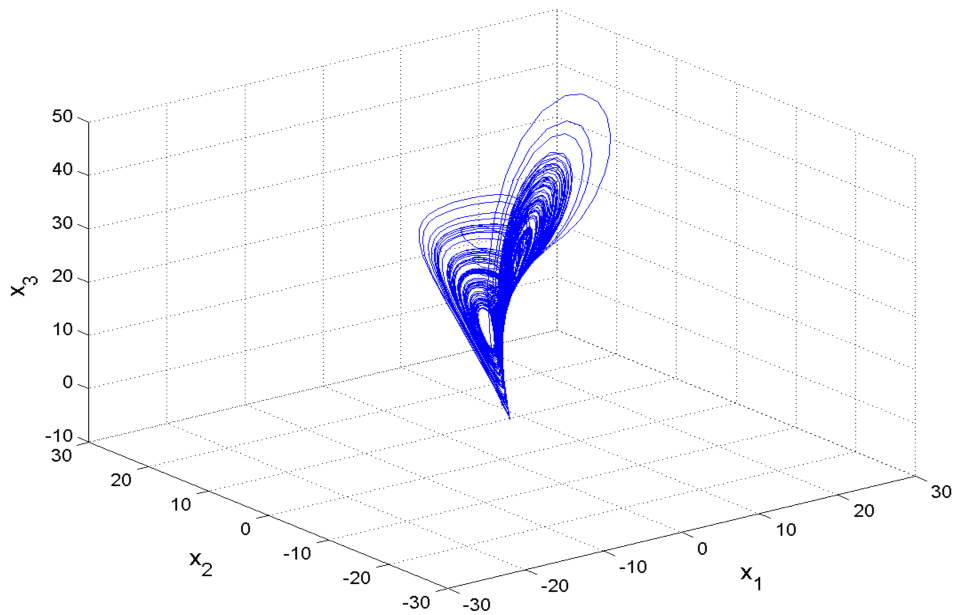


Figure 2. State Orbits of the Lü Chaotic System

3. Hybrid Synchronization of Identical Liu Chaotic Systems via Adaptive Control

3.1 Theoretical Results

In this section, we discuss the hybrid synchronization of identical Liu chaotic systems ([43], 2004), where the parameters of the master and slave systems are unknown.
As the master system, we consider the Liu dynamics described by

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) \\ \dot{x}_2 &= bx_1 - x_1 x_3 \\ \dot{x}_3 &= -cx_3 + dx_1^2 \end{aligned} \tag{3}$$

where x_1, x_2, x_3 are the state variables and a, b, c, d are unknown, real ,constant parameters of the system.

As the slave system, we consider the controlled Liu dynamics described by

$$\begin{aligned} \dot{y}_1 &= a(y_2 - y_1) + u_1 \\ \dot{y}_2 &= by_1 - y_1 y_3 + u_2 \\ \dot{y}_3 &= -cy_3 + dy_1^2 + u_3 \end{aligned} \tag{4}$$

where y_1, y_2, y_3 are the state variables and u_1, u_2, u_3 are the nonlinear controllers to be designed.

The hybrid chaos synchronization error is defined by

$$\begin{aligned} e_1 &= y_1 - x_1 \\ e_2 &= y_2 + x_2 \\ e_3 &= y_3 - x_3 \end{aligned} \tag{5}$$

From the error equations (5), it is clear that one part of the two chaotic systems is completely synchronized (first and third states), while the other part is completely anti-synchronized (second states) so that complete synchronization (CS) and anti-synchronization (AS) coexist in the synchronization of the chaotic systems (3) and (4).

The error dynamics is easily obtained as

$$\begin{aligned} \dot{e}_1 &= a(e_2 - e_1 - 2x_2) + u_1 \\ \dot{e}_2 &= b(e_1 + 2x_1) - y_1 y_3 - x_1 x_3 + u_2 \\ \dot{e}_3 &= -ce_3 + d\left(y_1^2 - x_1^2\right) + u_3 \end{aligned} \tag{6}$$

Let us now define the adaptive control functions

$$\begin{aligned} u_1(t) &= -\hat{a}(e_2 - e_1 - 2x_2) - k_1 e_1 \\ u_2(t) &= -\hat{b}(e_1 + 2x_1) + y_1 y_3 + x_1 x_3 - k_2 e_2 \\ u_3(t) &= \hat{c}e_3 - \hat{d}\left(y_1^2 - x_1^2\right) - k_3 e_3 \end{aligned} \tag{7}$$

where $\hat{a}, \hat{b}, \hat{c}$ and $\hat{d}$ are estimates of a, b, c and d, respectively, and $k_i, (i = 1, 2, 3)$ are positive constants.

Substituting (7) into (6), the error dynamics simplifies to

$$\begin{aligned} \dot{e}_1 &= (a - \hat{a})(e_2 - e_1 - 2x_2) - k_1 e_1 \\ \dot{e}_2 &= (b - \hat{b})(e_1 + 2x_1) - k_2 e_2 \\ \dot{e}_3 &= -(c - \hat{c})e_3 + (d - \hat{d})(y_1^2 - x_1^2) - k_3 e_3 \end{aligned} \tag{8}$$

Let us now define the parameter estimation errors as

$$e_a = a - \hat{a}, \quad e_b = b - \hat{b}, \quad e_c = c - \hat{c} \quad \text{and} \quad e_d = d - \hat{d} \tag{9}$$

Substituting (9) into (8), we obtain the error dynamics as

$$\begin{aligned}\dot{e}_1 &= e_a(e_2 - e_1 - 2x_2) - k_1 e_1 \\ \dot{e}_2 &= e_b(e_1 + 2x_1) - k_2 e_2 \\ \dot{e}_3 &= -e_c e_3 + e_d(y_1^2 - x_1^2) - k_3 e_3\end{aligned} \tag{10}$$

For the derivation of the update law for adjusting the estimates of the parameters, the Lyapunov approach is used.

We consider the quadratic Lyapunov function defined by

$$V(e_1, e_2, e_3, e_a, e_b, e_c, e_d) = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_a^2 + e_b^2 + e_c^2 + e_d^2\right), \tag{11}$$

which is a positive definite function on R^7.

We also note that

$$\dot{e}_a = -\dot{\hat{a}}, \ \dot{e}_b = -\dot{\hat{b}}, \ \dot{e}_c = -\dot{\hat{c}} \quad \text{and} \quad \dot{e}_d = -\dot{\hat{d}} \tag{12}$$

Differentiating (11) along the trajectories of (10) and using (12), we obtain

$$\begin{aligned}\dot{V} = &-k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 + e_a\left[e_1(e_2 - e_1 - 2x_2) - \dot{\hat{a}}\right] + e_b\left[e_2(e_1 + 2x_1) - \dot{\hat{b}}\right] \\ &+ e_c\left[-e_3^2 - \dot{\hat{c}}\right] + e_d\left[e_3(y_1^2 - x_1^2) - \dot{\hat{d}}\right]\end{aligned} \tag{13}$$

In view of Eq. (13), the estimated parameters are updated by the following law:

$$\begin{aligned}\dot{\hat{a}} &= e_1(e_2 - e_1 - 2x_2) + k_4 e_a \\ \dot{\hat{b}} &= e_2(e_1 + 2x_1) + k_5 e_b \\ \dot{\hat{c}} &= -e_3^2 + k_6 e_c \\ \dot{\hat{d}} &= e_3\left(y_1^2 - x_1^2\right) + k_7 e_d\end{aligned} \tag{14}$$

where k_4, k_5, k_6 and k_7 are positive constants.

Substituting (14) into (13), we obtain

$$\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_a^2 - k_5 e_b^2 - k_6 e_c^2 - k_7 e_d^2 \tag{15}$$

which is a negative definite function on R^7.

Thus, by Lyapunov stability theory [45], it is immediate that the hybrid synchronization error $e_i, (i = 1, 2, 3)$ and the parameter estimation error e_a, e_b, e_c, e_d decay to zero exponentially with time.

Hence, we have proved the following result.

Theorem 1. *The identical Liu chaotic systems (3) and (4) with unknown parameters are globally and exponentially hybrid synchronized via the adaptive control law (7), where the update law for the parameter estimates is given by (14) and* $k_i, (i = 1, 2, \ldots, 7)$ *are positive constants. Also, the parameter estimates* $\hat{a}(t), \hat{b}(t), \hat{c}(t)$ *and* $\hat{d}(t)$ *exponentially converge to the original values of the parameters* a, b, c *and* d*, respectively, as* $t \to \infty$. ■

3.2 Numerical Results

For the numerical simulations, the fourth-order Runge-Kutta method with time-step $h = 10^{-6}$ is used to solve the 3-D chaotic systems (3) and (4) with the adaptive control law (14) and the parameter update law (14) using MATLAB.

We take

$$k_i = 4 \text{ for } i = 1, 2, \ldots, 7.$$

For the Liu chaotic systems (3) and (4), the parameter values are taken as

$$a = 10, \quad b = 40, \quad c = 2.5, \quad d = 4$$

Suppose that the initial values of the parameter estimates are

$$\hat{a}(0) = 4, \quad \hat{b}(0) = 12, \quad \hat{c}(0) = 8, \quad \hat{d}(0) = 10.$$

The initial values of the master system (3) are taken as

$$x_1(0) = 7, \quad x_2(0) = 11, \quad x_3(0) = 15$$

The initial values of the slave system (4) are taken as

$$y_1(0) = 30, \quad y_2(0) = 18, \quad y_3(0) = 26$$

Figure 3 depicts the hybrid-synchronization of the identical Liu chaotic systems (3) and (4). It may also be noted that the odd states of the two systems are completely synchronized, while the even states of the two systems are anti-synchronized.

Figure 4 shows that the estimated values of the parameters, viz. $\hat{a}(t), \hat{b}(t), \hat{c}(t)$ and $\hat{d}(t)$ converge exponentially to the system parameters

$$a = 10, \quad b = 40, \quad c = 2.5 \text{ and } d = 4$$

as t tends to infinity.

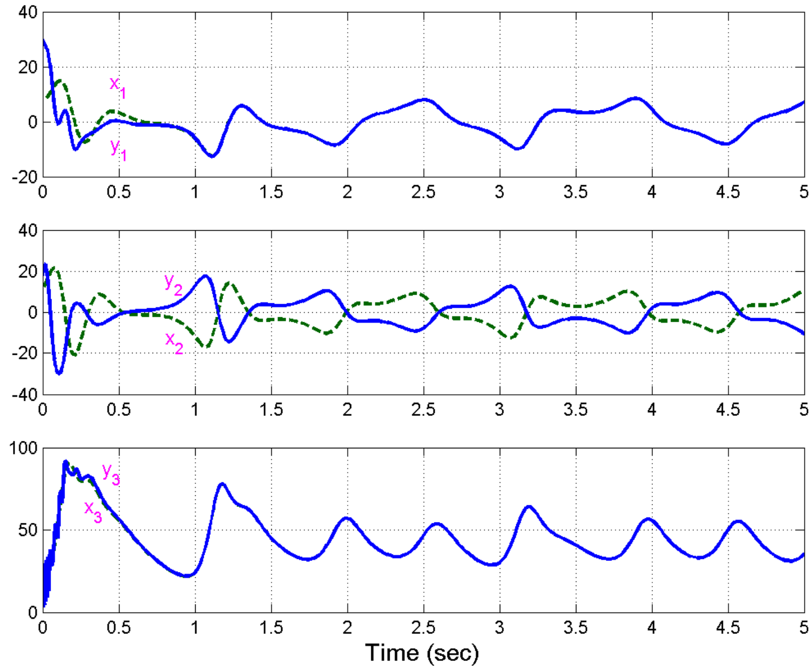


Figure 3. Hybrid-Synchronization of Liu Chaotic Systems

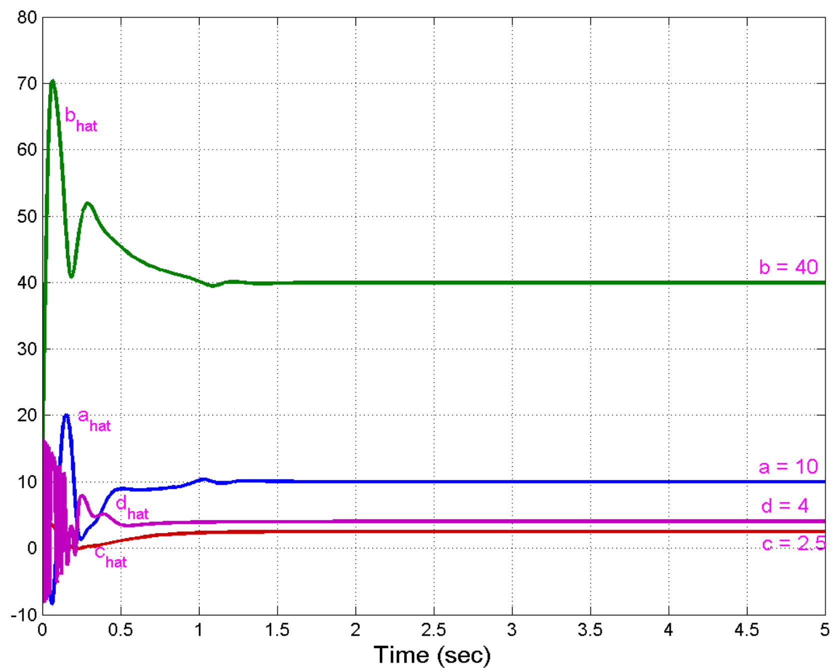


Figure 4. Parameter Estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{d}(t)$

4. HYBRID SYNCHRONIZATION OF IDENTICAL LÜ CHAOTIC SYSTEMS VIA ADAPTIVE CONTROL

4.1 Theoretical Results

In this section, we discuss the hybrid synchronization of identical Lü chaotic systems ([44], 2002), where the parameters of the master and slave systems are unknown.
As the master system, we consider the Lü dynamics described by

$$\begin{aligned} \dot{x}_1 &= \alpha(x_2 - x_1) \\ \dot{x}_2 &= \gamma x_2 - x_1 x_3 \\ \dot{x}_3 &= -\beta x_3 + x_1 x_2 \end{aligned} \tag{16}$$

where x_1, x_2, x_3 are the state variables and α, β, γ are unknown, real ,constant parameters of the system.

As the slave system, we consider the controlled Lü dynamics described by

$$\begin{aligned} \dot{y}_1 &= \alpha(y_2 - y_1) + u_1 \\ \dot{y}_2 &= \gamma y_2 - y_1 y_3 + u_2 \\ \dot{y}_3 &= -\beta y_3 + y_1 y_2 + u_3 \end{aligned} \tag{17}$$

where y_1, y_2, y_3 are the state variables and u_1, u_2, u_3 are the nonlinear controllers to be designed.

The hybrid chaos synchronization error is defined by

$$\begin{aligned} e_1 &= y_1 - x_1 \\ e_2 &= y_2 + x_2 \\ e_3 &= y_3 - x_3 \end{aligned} \tag{18}$$

From the error equations (18), it is clear that one part of the two chaotic systems is completely synchronized (first and third states), while the other part is completely anti-synchronized (second states) so that complete synchronization (CS) and anti-synchronization (AS) coexist in the synchronization of the chaotic systems (16) and (17).

The error dynamics is easily obtained as

$$\begin{aligned} \dot{e}_1 &= \alpha(e_2 - e_1 - 2x_2) + u_1 \\ \dot{e}_2 &= \gamma e_2 - y_1 y_3 - x_1 x_3 + u_2 \\ \dot{e}_3 &= -\beta e_3 + y_1 y_2 - x_1 x_2 + u_3 \end{aligned} \tag{19}$$

Let us now define the adaptive control functions

$$\begin{aligned} u_1(t) &= -\hat{\alpha}(e_2 - e_1 - 2x_2) - k_1 e_1 \\ u_2(t) &= -\hat{\gamma} e_2 + y_1 y_3 + x_1 x_3 - k_2 e_2 \\ u_3(t) &= \hat{\beta} e_3 - y_1 y_2 + x_1 x_2 - k_3 e_3 \end{aligned} \tag{20}$$

where $\hat{\alpha}, \hat{\beta}$ and $\hat{\gamma}$ are estimates of α, β and γ, respectively, and $k_i, (i = 1, 2, 3)$ are positive constants.

Substituting (20) into (19), the error dynamics simplifies to

$$\begin{aligned} \dot{e}_1 &= (\alpha - \hat{\alpha})(e_2 - e_1 - 2x_2) - k_1 e_1 \\ \dot{e}_2 &= (\gamma - \hat{\gamma}) e_2 - k_2 e_2 \\ \dot{e}_3 &= -(\beta - \hat{\beta}) e_3 - k_3 e_3 \end{aligned} \tag{21}$$

Let us now define the parameter estimation errors as

$$\begin{aligned} e_\alpha &= \alpha - \hat{\alpha} \\ e_\beta &= \beta - \hat{\beta} \\ e_\gamma &= \gamma - \hat{\gamma} \end{aligned} \tag{22}$$

Substituting (22) into (21), we obtain the error dynamics as

$$\begin{aligned} \dot{e}_1 &= e_\alpha (e_2 - e_1 - 2x_2) - k_1 e_1 \\ \dot{e}_2 &= e_\gamma e_2 - k_2 e_2 \\ \dot{e}_3 &= -e_\beta e_3 - k_3 e_3 \end{aligned} \tag{23}$$

For the derivation of the update law for adjusting the estimates of the parameters, the Lyapunov approach is used.

We consider the quadratic Lyapunov function defined by

$$V(e_1, e_2, e_3, e_\alpha, e_\beta, e_\gamma) = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_\alpha^2 + e_\beta^2 + e_\gamma^2\right), \tag{24}$$

which is a positive definite function on R^6.

We also note that

$$\begin{aligned} \dot{e}_\alpha &= -\dot{\hat{\alpha}} \\ \dot{e}_\beta &= -\dot{\hat{\beta}} \\ \dot{e}_\gamma &= -\dot{\hat{\gamma}} \end{aligned} \tag{25}$$

Differentiating (24) along the trajectories of (23) and using (25), we obtain

$$\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 + e_\alpha \left[e_1 (e_2 - e_1 - 2x_2) - \dot{\hat{\alpha}} \right] + e_\beta \left[-e_3^2 - \dot{\hat{\beta}} \right] + e_\gamma \left[e_2^2 - \dot{\hat{\gamma}} \right] \tag{26}$$

In view of Eq. (26), the estimated parameters are updated by the following law:

$$\begin{aligned} \dot{\hat{\alpha}} &= e_1 (e_2 - e_1 - 2x_2) + k_4 e_\alpha \\ \dot{\hat{\beta}} &= -e_3^2 + k_5 e_\beta \\ \dot{\hat{\gamma}} &= e_2^2 + k_6 e_\gamma \end{aligned} \tag{27}$$

where k_4, k_5 and k_6 are positive constants.

Substituting (27) into (26), we obtain

$$\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_\alpha^2 - k_5 e_\beta^2 - k_6 e_\gamma^2 \tag{28}$$

which is a negative definite function on R^6.

Thus, by Lyapunov stability theory [45], it is immediate that the hybrid synchronization error $e_i, (i = 1, 2, 3)$ and the parameter estimation error $e_\alpha, e_\beta, e_\gamma$ decay to zero exponentially with time.

Hence, we have proved the following result.

Theorem 2. *The identical Lü chaotic systems (16) and (17) with unknown parameters are globally and exponentially hybrid synchronized via the adaptive control law (20), where the update law for the parameter estimates is given by (27) and* $k_i, (i = 1, 2, \ldots, 6)$ *are positive constants. Also, the parameter estimates* $\hat{\alpha}(t), \hat{\beta}(t)$ *and* $\hat{\gamma}(t)$ *exponentially converge to the original values of the parameters* α, β *and* γ, *respectively, as* $t \to \infty$. ■

4.2 Numerical Results

For the numerical simulations, the fourth-order Runge-Kutta method with time-step $h = 10^{-6}$ is used to solve the 3-D chaotic systems (16) and (17) with the adaptive control law (20) and the parameter update law (27) using MATLAB.

We take $k_i = 4$ for $i = 1, 2, \ldots, 6$.

For the Lü chaotic systems (16) and (17), the parameter values are taken as

$$\alpha = 36, \quad \beta = 3, \quad \gamma = 20$$

Suppose that the initial values of the parameter estimates are

$$\hat{\alpha}(0) = 9, \quad \hat{\beta}(0) = 24, \quad \hat{\gamma}(0) = 11$$

The initial values of the master system (16) are taken as

$$x_1(0) = 22, \quad x_2(0) = 14, \quad x_3(0) = 30$$

The initial values of the slave system (17) are taken as

$$y_1(0) = 10, \quad y_2(0) = 28, \quad y_3(0) = 7$$

Figure 5 depicts the hybrid-synchronization of the identical Lü chaotic systems (16) and (17). It may also be noted that the odd states of the two systems are completely synchronized, while the even states of the two systems are anti-synchronized.

Figure 6 shows that the estimated values of the parameters, viz. $\hat{\alpha}(t), \hat{\beta}(t)$ and $\hat{\gamma}(t)$ converge exponentially to the system parameters $\alpha = 36, \quad \beta = 3$ and $\gamma = 20,$ respectively, as $t \to \infty.$

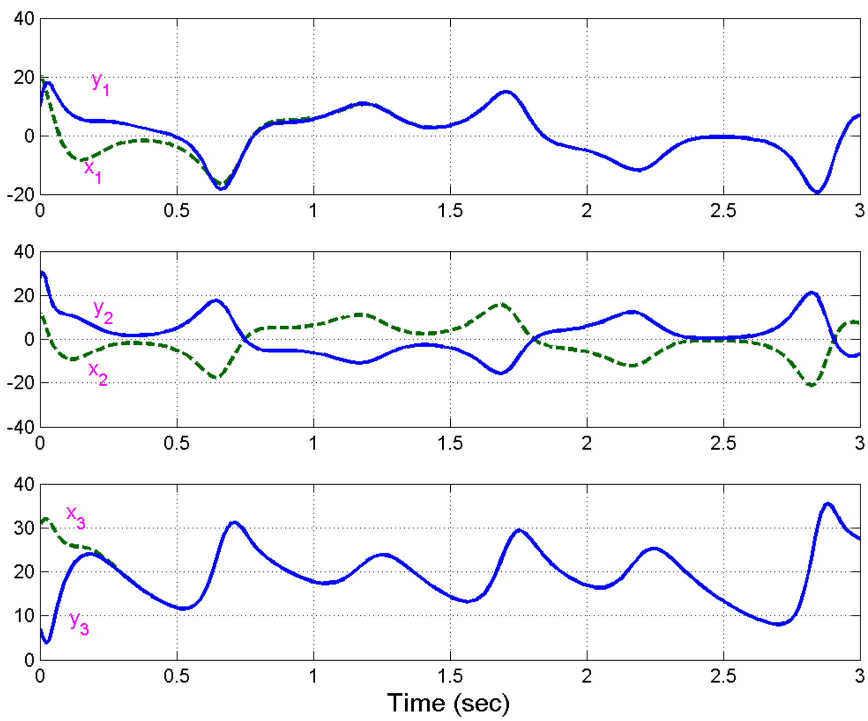
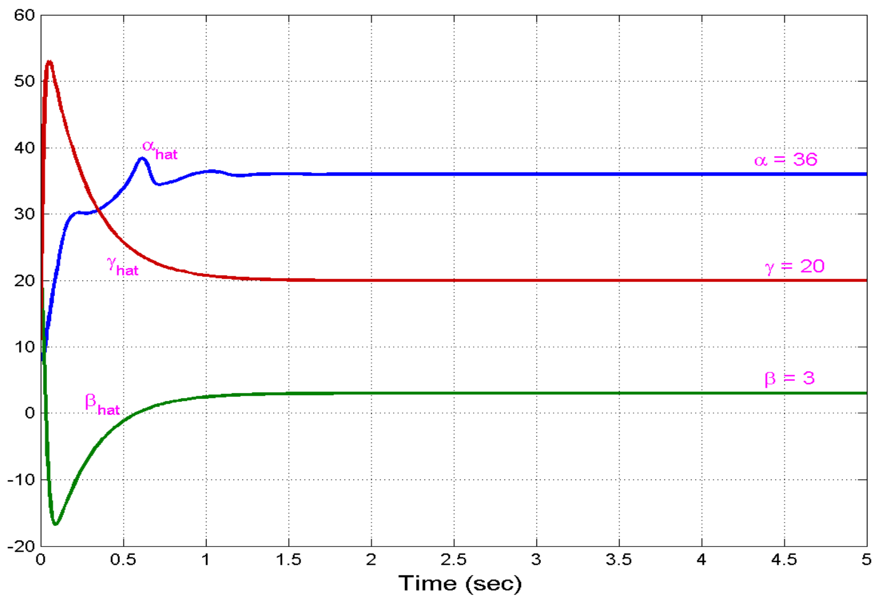


Figure 5. Hybrid-Synchronization of Lü Chaotic Systems



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5. Hybrid Synchronization of Liu and Lü Chaotic Systems via Adaptive Control

5.1 Theoretical Results

In this section, we discuss the hybrid synchronization of identical Lü chaotic systems ([44], 2002), where the parameters of the master and slave systems are unknown.
As the master system, we consider the Liu dynamics described by

$$\begin{aligned}\dot{x}_1 &= a(x_2 - x_1)\\ \dot{x}_2 &= bx_1 - x_1x_3\\ \dot{x}_3 &= -cx_3 + dx_1^2\end{aligned} \tag{29}$$

where x_1, x_2, x_3 are the state variables and a, b, c, d are unknown, real ,constant parameters of the system.

As the slave system, we consider the controlled Lü dynamics described by

$$\begin{aligned}\dot{y}_1 &= \alpha(y_2 - y_1) + u_1\\ \dot{y}_2 &= \gamma y_2 - y_1y_3 + u_2\\ \dot{y}_3 &= -\beta y_3 + y_1y_2 + u_3\end{aligned} \tag{30}$$

where y_1, y_2, y_3 are the state variables , α, β, γ are unknown, real, constant parameters of the system and u_1, u_2, u_3 are the nonlinear controllers to be designed.

The hybrid chaos synchronization error is defined by

$$\begin{aligned} e_1 &= y_1 - x_1 \\ e_2 &= y_2 + x_2 \\ e_3 &= y_3 - x_3 \end{aligned} \tag{31}$$

From the error equations (31), it is clear that one part of the two chaotic systems is completely synchronized (first and third states), while the other part is completely anti-synchronized (second states) so that complete synchronization (CS) and anti-synchronization (AS) coexist in the synchronization of the chaotic systems (29) and (30).

The error dynamics is easily obtained as

$$\begin{aligned} \dot{e}_1 &= \alpha(y_2 - y_1) - a(x_2 - x_1) + u_1 \\ \dot{e}_2 &= \gamma y_2 + b x_1 - y_1 y_3 - x_1 x_3 + u_2 \\ \dot{e}_3 &= -\beta y_3 + c x_3 + y_1 y_2 - d x_1^2 + u_3 \end{aligned} \tag{32}$$

Let us now define the adaptive control functions

$$\begin{aligned} u_1(t) &= -\hat{\alpha}(y_2 - y_1) + \hat{a}(x_2 - x_1) - k_1 e_1 \\ u_2(t) &= -\hat{\gamma} y_2 - \hat{b} x_1 + y_1 y_3 + x_1 x_3 - k_2 e_2 \\ u_3(t) &= \hat{\beta} y_3 - \hat{c} x_3 + \hat{d} x_1^2 - y_1 y_2 - k_3 e_3 \end{aligned} \tag{33}$$

where $\hat{a}, \hat{b}, \hat{c}, \hat{d}$, $\hat{\alpha}, \hat{\beta}$ and $\hat{\gamma}$ are estimates of a, b, c, d, α, β and γ, respectively, and $k_i, (i = 1, 2, 3)$ are positive constants.

Substituting (33) into (32), the error dynamics simplifies to

$$\begin{aligned} \dot{e}_1 &= (\alpha - \hat{\alpha})(y_2 - y_1) - (a - \hat{a})(x_2 - x_1) - k_1 e_1 \\ \dot{e}_2 &= (\gamma - \hat{\gamma}) y_2 + (b - \hat{b}) x_1 - k_2 e_2 \\ \dot{e}_3 &= -(\beta - \hat{\beta}) y_3 + (c - \hat{c}) x_3 - (d - \hat{d}) x_1^2 - k_3 e_3 \end{aligned} \tag{34}$$

Let us now define the parameter estimation errors as

$$\begin{aligned} & e_a = a - \hat{a}, \;\; e_b = b - \hat{b}, \;\; e_c = c - \hat{c}, \;\; e_d = d - \hat{d} \\ & e_\alpha = \alpha - \hat{\alpha}, \;\; e_\beta = \beta - \hat{\beta}, \;\; e_\gamma = \gamma - \hat{\gamma} \end{aligned} \tag{35}$$

Substituting (35) into (34), we obtain the error dynamics as

$$\begin{aligned} \dot{e}_1 &= e_\alpha (y_2 - y_1) - e_a (x_2 - x_1) - k_1 e_1 \\ \dot{e}_2 &= e_\gamma y_2 + e_b x_1 - k_2 e_2 \\ \dot{e}_3 &= -e_\beta y_3 + e_c x_3 - e_d x_1^2 - k_3 e_3 \end{aligned} \tag{36}$$

For the derivation of the update law for adjusting the estimates of the parameters, the Lyapunov approach is used.

We consider the quadratic Lyapunov function defined by

$$V = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_a^2 + e_b^2 + e_c^2 + e_d^2 + e_\alpha^2 + e_\beta^2 + e_\gamma^2\right), \tag{37}$$

which is a positive definite function on R^{10}.

We also note that

$$\begin{aligned} &\dot{e}_a = -\dot{\hat{a}}, \;\; \dot{e}_b = -\dot{\hat{b}}, \;\; \dot{e}_c = -\dot{\hat{c}}, \;\; \dot{e}_d = -\dot{\hat{d}} \\ &\dot{e}_\alpha = -\dot{\hat{\alpha}}, \; \dot{e}_\beta = -\dot{\hat{\beta}}, \; \dot{e}_\gamma = -\dot{\hat{\gamma}} \end{aligned} \tag{38}$$

Differentiating (37) along the trajectories of (36) and using (38), we obtain

$$\begin{aligned} \dot{V} = &-k_1e_1^2 - k_2e_2^2 - k_3e_3^2 + e_a\left[-e_1(x_2 - x_1) - \dot{\hat{a}}\right] + e_b\left[e_2x_1 - \dot{\hat{b}}\right] + e_c\left[e_3x_3 - \dot{\hat{c}}\right] \\ &+ e_d\left[-e_3x_1^2 - \dot{\hat{d}}\right] + e_\alpha\left[e_1(y_2 - y_1) - \dot{\hat{\alpha}}\right] + e_\beta\left[-e_3y_3 - \dot{\hat{\beta}}\right] + e_\gamma\left[e_2y_2 - \dot{\hat{\gamma}}\right] \end{aligned} \tag{39}$$

In view of Eq. (39), the estimated parameters are updated by the following law:

$$\begin{aligned} &\dot{\hat{a}} = -e_1(x_2 - x_1) + k_4e_a, \qquad &&\dot{\hat{\alpha}} = e_1(y_2 - y_1) + k_8e_\alpha \\ &\dot{\hat{b}} = e_2x_1 + k_5e_b, &&\dot{\hat{\beta}} = -e_3y_3 + k_9e_\beta \\ &\dot{\hat{c}} = e_3x_3 + k_6e_c, &&\dot{\hat{\gamma}} = e_2y_2 + k_{10}e_\gamma \\ &\dot{\hat{d}} = -e_3x_1^2 + k_7e_d && \end{aligned} \tag{40}$$

where $k_i, (i = 4, \ldots, 10)$ are positive constants.

Substituting (40) into (39), we obtain

$$\dot{V} = -k_1e_1^2 - k_2e_2^2 - k_3e_3^2 - k_4e_a^2 - k_5e_b^2 - k_6e_c^2 - k_7e_d^2 - k_8e_\alpha^2 - k_9e_\beta^2 - k_{10}e_\gamma^2 \tag{41}$$

which is a negative definite function on R^{10}.

Thus, by Lyapunov stability theory [45], it is immediate that the hybrid synchronization error $e_i, (i = 1, 2, 3)$ and the parameter estimation error $e_a, e_b, e_c, e_d, e_\alpha, e_\beta, e_\gamma$ decay to zero exponentially with time.

Hence, we have proved the following result.

Theorem 3. *The non-identical Liu system (29) and Lü system (30) with unknown parameters are globally and exponentially hybrid synchronized via the adaptive control law (33), where the* *update law for the parameter estimates is given by (40) and* $k_i, (i = 1, 2, \ldots, 10)$ *are positive constants. Also, the parameter estimates* $\hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{d}(t), \hat{\alpha}(t), \hat{\beta}(t)$ *and* $\hat{\gamma}(t)$ *exponentially converge to the original values of the parameters* $a, b, c, d, \alpha, \beta$ *and* γ, *respectively, as* $t \to \infty$. ■

5.2 Numerical Results

For the numerical simulations, the fourth-order Runge-Kutta method with time-step $h = 10^{-6}$ is used to solve the 3-D chaotic systems (29) and (30) with the adaptive control law (33) and the parameter update law (40) using MATLAB.

We take $k_i = 4$ for $i = 1, 2, \ldots, 10$.

For the Liu chaotic system, the parameter values are taken as

$$a = 10, \; b = 40, \; c = 2.5, \quad d = 4$$

For the Lü chaotic system, the parameter values are taken as

$$\alpha = 36, \quad \beta = 3, \quad \gamma = 20$$

Suppose that the initial values of the parameter estimates are

$$\hat{a}(0) = 7, \; \hat{b}(0) = 15, \; \hat{c}(0) = 20, \; \hat{d}(0) = 9, \; \hat{\alpha}(0) = 12, \quad \hat{\beta}(0) = 6, \; \hat{\gamma}(0) = 5$$

The initial values of the master system (29) are taken as

$$x_1(0) = 17, \;\; x_2(0) = 11, \;\; x_3(0) = 28$$

The initial values of the slave system (30) are taken as

$$y_1(0) = 30, \;\; y_2(0) = 16, \;\; y_3(0) = 12$$

Figure 7 depicts the hybrid-synchronization of the non-identical Liu system (29) and Lü chaotic system (30). It may also be noted that the odd states of the two systems are completely synchronized, while the even states of the two systems are anti-synchronized.

Figure 8 shows that the estimated values of the parameters, viz. $\hat{a}(t)$, $\hat{b}(t)$, $\hat{c}(t)$, $\hat{d}(t)$, $\hat{\alpha}(t)$, $\hat{\beta}(t)$ and $\hat{\gamma}(t)$ converge exponentially to the system parameters $a = 10$, $b = 40$, $c = 2.5$, $d = 4$, $\alpha = 36$, $\beta = 3$ and $\gamma = 20$, respectively, as $t \to \infty$.

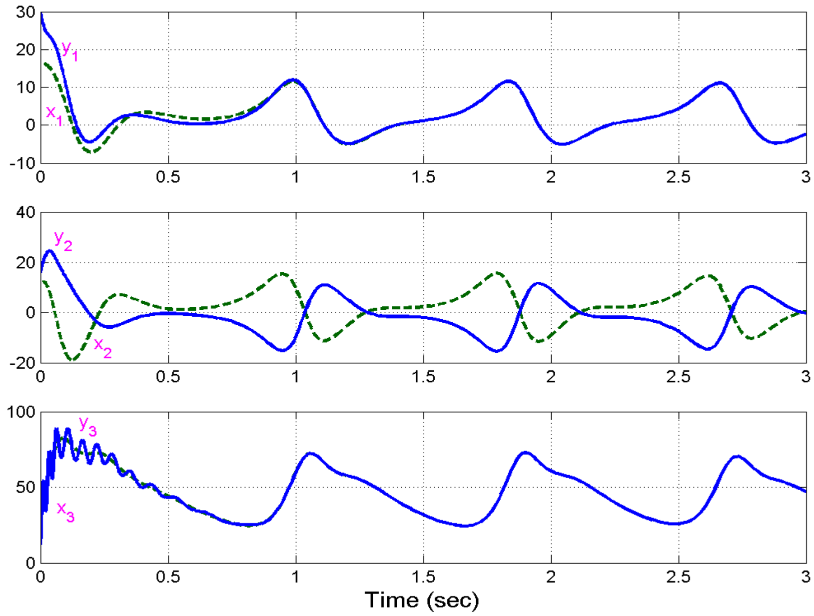

Figure 7. Hybrid-Synchronization of Liu and Lü Chaotic Systems

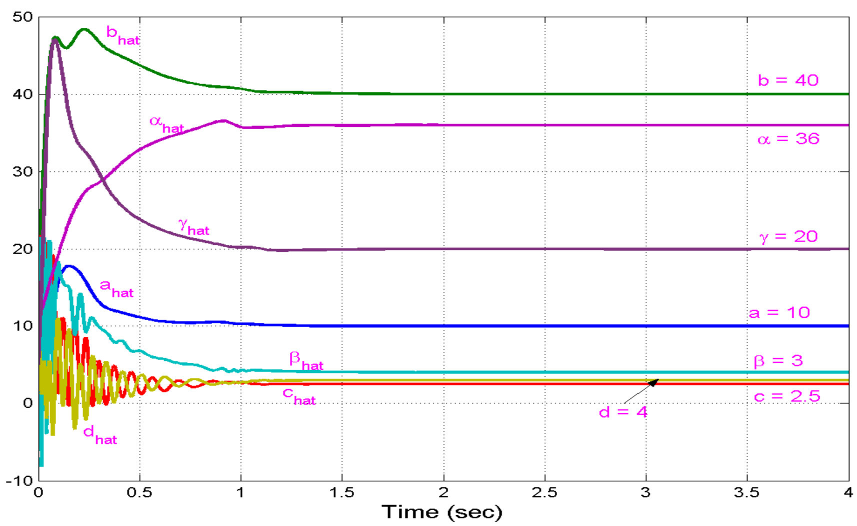

Figure 8. Parameter Estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{d}(t), \hat{\alpha}(t), \hat{\beta}(t), \hat{\gamma}(t)$

6. CONCLUSIONS

In this paper, we have deployed adaptive control method to derive new results for the hybrid synchronization of identical Liu systems (2004), identical Lü systems (2002) and non-identical Liu and Lü systems with unknown parameters. The hybrid synchronization results derived in this paper are established using Lyapunov stability theory. Since the Lyapunov exponents are not required for these calculations, the adaptive control method is a very effective and convenient for achieving hybrid chaos synchronization for the uncertain three-dimensional chaotic systems discussed in this paper. Numerical simulations are presented to demonstrate the effectiveness of the adaptive synchronization schemes derived in this paper for the hybrid chaos synchronization of identical and non-identical uncertain Liu and Lü systems.

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